JOURNAL OF APPROXIMATION THEORY 26, 132–141 (1979)

On n-Widths and Interpolation by Polynomial Splines

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Communicated by E. W. Cheney

Received December 19, 1977

1. INTRODUCTION

We consider the following quasi-Hermite interpolation problem. Let n, r, k, z_j, q_{0i} $(0 \leqslant i \leqslant z_0 - 1)$, q_{1i} $(0 \leqslant i \leqslant z_{k+1} - 1)$ be given integers such that $n \geqslant 0$, $r \geqslant 2$, $k \geqslant 0$, $1 \leqslant z_j \leqslant r - 1$ for $1 \leqslant j \leqslant k$, $0 \leqslant z_j \leqslant r - 1$ for $j = 0$ and $j = k + 1$, $0 \leqslant q_{0,0} < \cdots < q_{0,z_0-1} \leqslant r - 2$, and $0 \leqslant q_{1,0} < \cdots < q_{1,z_{k+1}-1} \leqslant r - 2$. Let also $k + 2$ abscissae x_j be given such that

$$0 = x_0 < x_1 < \cdots < x_{k+1} = 1.$$

DEFINITION 1. We say that a function $g \in C^{r-2}[0, 1]$ interpolates $f \in C^{r-2}[0, 1]$, provided that

$$\begin{aligned}
g^{(q)}(x_j) &= f^{(q)}(x_j) && \text{for } 0 \leqslant q \leqslant z_j - 1, \quad 1 \leqslant j \leqslant k, \\
g^{(q_{0i})}(0) &= f^{(q_{0i})}(0) && \text{for } 0 \leqslant i \leqslant z_0 - 1 \text{ (no condition if } z_0 = 0), \qquad (1) \\
g^{(q_{1i})}(1) &= f^{(q_{1i})}(1) && \text{for } 0 \leqslant i \leqslant z_{k+1} - 1 \text{ (no condition if } z_{k+1} = 0).
\end{aligned}$$

Throughout this paper we shall interpolate by polynomial splines.

DEFINITION 2. Let

$$\Delta: 0 = t_0 < t_1 < \cdots < t_{n+1} = 1 \tag{2}$$

be a collection of knots. The set of polynomial splines of degree $r - 1$ with knots Δ is

$$\mathrm{Sp}(r, \Delta) = \left\{ s \in C^{r-2}[0, 1]: s(x) = \sum_{i=0}^{r-1} a_i x^i + \sum_{j=1}^{n} c_j (x - t_j)_+^{r-1} \right\},$$

* Supported by Deutsche Forschungsgemeinschaft, Grant Go 270/2.

0021-9045/79/060132–10$02.00/0

where a_i and c_j are any real numbers and $x_+^{r-1} = x^{r-1}$ if $x > 0$, $x_+^{r-1} = 0$ if $x \leqslant 0$.

We shall interpolate functions $f \in C^{r-2}[0, 1]$ and, in particular, functions belonging to the following classes.

DEFINITION 3. If Δ denotes a collection (2) of knots, then

$$W_{r,\Delta} = \{f \in C^{r-2}[0, 1]: f^{(r-1)} \text{ is absolutely continuous in each open interval } (t_j, t_{j+1}),\ 0 \leqslant j \leqslant n\}.$$

If $w \in L^1[0, 1]$, then

$$W_{r,w,\Delta} = \{f \in W_{r,\Delta}: |f^{(r)}(x)| \leqslant |w(x)| \text{ a.e. in } [0, 1]\}.$$

In particular

$$W_{r,1,\Delta} = \{f \in W_{r,\Delta}: |f^{(r)}(x)| \leqslant 1 \text{ a.e. in } [0, 1]\}.$$

In Section 2 we discuss necessary and sufficient conditions under which the interpolation problem (1) has a unique spline interpolant $s_f \in \mathrm{Sp}(r, \Delta)$ for every function $f \in C^{r-2}[0, 1]$.

If the interpolation problem (1) satisfies these conditions then Theorem 2 in Section 3 states that the classes $W_{r,w,\Delta}$ possess a unique extremal function $F \in W_{r,w,\Delta}$ provided that $w \in L^1[0, 1]$ and $|w(x)| > 0$ a.e. in $[0, 1]$. The functions F are even locally extremal, i.e.

$$\sup\{|f(x) - s_f(x)| : f \in W_{r,w,\Delta}\} = |F(x)| \qquad \text{for every} \quad x \in [0, 1].$$

In Section 4 we study the classes of functions $W_r := \{f \mid f^{(r-1)}$ absolutely continuous on $[0, 1]$, $|f^{(r)}(x)| \leqslant 1$ a.e. on $[0, 1]\}$ and the splines $\mathrm{Sp}(r, \Delta^*)$ with equidistant knots Δ^*. If we choose x_j to be the knots Δ^* (if r is even) and to be the midpoints between the knots Δ^* (if r is odd) and choose suitable interpolation conditions at the endpoints $x_0 = 0$ and $x_{k+1} = 1$, then Theorem 2 leads to the inequality

$$\sup\{\|f - s_f\|_{C[0,1]} : f \in W_r\} \leqslant d_n(W_r)$$

where $d_n(W_r)$ denotes the n-width of W_r in $C[0, 1]$. Since $\dim \mathrm{Sp}(r, \Delta^*) = n + r$ we have obtained the interesting result that the above described interpolation by polynomial splines $\mathrm{Sp}(r, \Delta^*)$ with equidistant knots leads to asymptotically best possible error bounds in the sense of n-widths.

In Section 5 the application of Theorem 2 to cubic splines with arbitrary knots Δ leads to some new local error bounds.

2. Existence and Uniqueness in Spline Interpolation

There is a series of papers on the existence and uniqueness in spline interpolation for more general interpolation problems than (1) (see Schoenberg [9], Karlin [4], Karlin and Karon [5], Melkman [6], a.o.). In particular, the following theorem is a special case of Melkman [6; Theorem 1].

THEOREM 1. *Let* Δ *be a collection* (2) *of* n *knots in* $(0, 1)$. *Assume that the interpolation problem* (1) *contains* $n + r$ *conditions, i.e.*

$$\sum_{j=0}^{k+1} z_j = n + r. \tag{3}$$

Let m_q $(q = 0, 1,..., r - 2)$ *denote the number of interpolation conditions* (1) *for the q-th derivative* $f^{(q)}$ *in* $[0, 1]$, *and let* L_j *and* U_j $(j = 1,..., n)$ *denote the number of interpolation conditions* (1) *in* $[0, t_j)$ *and* $(t_j, 1]$, *respectively. Then the spline interpolant* $s_f \in \mathrm{Sp}(r, \Delta)$ *exists and is unique for every function* $f \in C^{r-2}[0, 1]$ *if and only if Polya's conditions*

$$\sum_{q=0}^{p} m_q \geqslant p + 1, \qquad p = 0, 1,..., r - 2 \tag{4}$$

and the inequalities

$$L_j \geqslant j, \qquad U_j \geqslant n + 1 - j, \qquad j = 1,..., n \tag{5}$$

are satisfied.

3. Local Upper Bounds in Spline Interpolation

One of our main results is the following theorem. It provides best possible local upper bounds in polynomial spline interpolation for classes $W_{r,w,\Delta}$ of functions.

THEOREM 2. *Let a quasi-Hermite interpolation problem* (1) *and a collection* (2) *of knots* Δ *be given such that the assumptions* (3)–(5) *hold.*

(a) *For each function* $w \in L^1[0, 1]$ *there exists a unique function* $F \in W_{r,w,\Delta}$ *with the properties*

$$\begin{aligned} &\text{(i)} \quad F \text{ interpolates the zero function (see Definition 1).} \\ &\text{(ii)} \quad F^{(r)}(x) = (-1)^j\, |w(x)| \text{ a.e. in } (t_j, t_{j+1}),\ 0 \leqslant j \leqslant n. \end{aligned} \tag{6}$$

(b) *If* $|w(x)| > 0$ *a.e. on* $[0, 1]$, *for each* $f \in W_{r,w,\Delta}$ *the unique spline interpolant* $s_f \in \mathrm{Sp}(r, \Delta)$ *satisfies the inequalities*

$$|f(x) - s_f(x)| \leqslant |F(x)| \qquad \text{for all } x \in [0, 1] \tag{7}$$

$$|f^{(z_j)}(x_j) - s_f^{(z_j)}(x_j)| \leqslant |F^{(z_j)}(x_j)| \qquad \text{for } 1 \leqslant j \leqslant k \text{ if } z_j \leqslant r-2 \tag{8}$$

$$|f^{(q)}(x_j) - s_f^{(q)}(x_j)| \leqslant |F^{(q)}(x_j)| \qquad \text{for } j = 0, j = k+1 \text{ and all } 0 \leqslant q \leqslant r-2. \tag{9}$$

Remark 1. Since F has the spline interpolant $s_F \equiv 0$ it follows from (7) that

$$\sup\{|f(x) - s_f(x)| : f \in W_{r,w,\Delta}\} = |F(x)| \qquad \text{for each} \quad x \in [0, 1],$$

which means that F is extremal in $W_{r,w,\Delta}$ for each x.

Proof of Theorem 2. (a) Let $u \in L^1[0, 1]$ be defined by $u(x) = (-1)^j |w(x)|$ for $x \in (t_j, t_{j+1})$, $0 \leqslant j \leqslant n$. Let G be any r-th integral of u in $[0, 1]$. Then $G \in W_{r,w,\Delta}$ and G has a unique spline interpolant $s_G \in \mathrm{Sp}(r, \Delta)$. The function $F := G - s_G$ has the desired properties (6) and, of course, $F \in W_{r,w,\Delta}$. If $F_1 \in W_{r,w,\Delta}$ has also the properties (6), then $F - F_1 \in \mathrm{Sp}(r, \Delta)$ interpolates the zero function and is therefore itself identically zero. Hence F is unique.

(b) Assume there exists a number $z \in [0, 1]$ such that

$$|f(z) - s_f(z)| > |F(z)|.$$

We set $\Phi := F(z)/(f(z) - s_f(z))$ and consider the function $H := F - \Phi(f - s_f)$ which obviously has the following properties. $H \in W_{r,\Delta}$ because $F, f, s_f \in W_{r,\Delta}$,

$$(-1)^j H^{(r)}(x) > 0 \text{ a.e. in } (t_j, t_{j+1}), \qquad 0 \leqslant j \leqslant n, \tag{10}$$

because $|\Phi| < 1$ and $|w(x)| > 0$ a.e. on $[0, 1]$,

$$H \text{ interpolates the zero function and, additionally, } H(z) = 0. \tag{11}$$

Hence H has at least $m_0 + 1$ distinct zeros in $[0, 1]$. By Rolle's theorem, the first derivative H' has at least m_0 distinct zeros in $(0, 1)\backslash\{x_1;...; x_k\}$ and at least $m_0 + m_1$ distinct zeros in $[0, 1]$. By the same argument we prove that the second derivative H'' has at least $m_0 + m_1 + m_2 - 1$ distinct zeros in $[0, 1]$ and, by induction, that $H^{(r-2)}$ has at least $\sum_{q=0}^{r-2} m_q - (r-3) = n+3$ distinct zeros in $[0, 1]$.

However, since (10) holds and $H^{(r-2)} \in C[0, 1]$, it is easy to verify that $H^{(r-2)}$ can have at most $n + 2$ distinct zeros in $[0, 1]$. This contradiction proves the inequalities (7). (8) is an immediate consequence of (7). The proof for (9) is the same as for (7).

4. On n-Widths and Interpolation by Splines with Equidistant Knots

As our first application of Theorem 2 we consider the following special case of the interpolation problem (1): Let $n \geqslant 0$, $r \geqslant 2$, Δ^* be the collection of equidistant knots $t_j = j/(n+1)$, i.e.,

$$\Delta^*: 0 = t_0 < \cdots < t_j = \frac{j}{n+1} < \cdots < t_{n+1} = 1. \tag{12}$$

We say that $g \in C^{r-2}[0, 1]$ interpolates $f \in C^{r-2}[0, 1]$ if

$$g(t_j) = f(t_j) \quad \text{for} \quad 0 \leqslant j \leqslant n+1, \qquad g^{(2q)}(t_j) = f^{(2q)}(t_j) \tag{13a}$$

for $j = 0$ and $j = n+1$, $1 \leqslant q \leqslant (r-2)/2$, provided that r is even, and

$$g\left(\frac{2j+1}{2n+2}\right) = f\left(\frac{2j+1}{2n+2}\right) \text{ for } 0 \leqslant j \leqslant n, \qquad g^{(2q-1)}(t_j) = f^{(2q-1)}(t_j) \tag{13b}$$

for $j = 0$ and $j = n+1$, $1 \leqslant q \leqslant (r-1)/2$, provided that r is odd.

Obviously, the interpolation problem (13) satisfies the assumptions (3)–(5) of Theorem 2 which therefore can be applied to each weight function $w \in L^1[0, 1]$ with $|w(x)| > 0$ a.e. on $[0, 1]$. In this section we want to analyse the weight function $w \equiv 1$, for which we are able to present the extremal function F of Theorem 2 explicitly as follows. Let E_r be the Euler polynomial of degree r defined by the relation

$$E_r(x) + E_r(x+1) = 2x^r/r! \tag{14}$$

It is known that

$$\| E_r \| = K_r \pi^{-r}, \qquad K_r = \frac{4}{\pi} \sum_{k=0}^{\infty} \frac{(-1)^{k(r+1)}}{(2k+1)^{r+1}} \tag{15}$$

where $\| \cdot \| = \| \cdot \|_{C[0,1]}$ denotes the supremum norm in $[0, 1]$. Moreover,

$$E_r^{(2q)}(0) = E_r^{(2q)}(1) = 0 \qquad \text{for} \quad 0 \leqslant q \leqslant (r-2)/2 \tag{16a}$$

provided that r is even, and

$$E_r(\tfrac{1}{2}) = 0, \qquad E_r^{(2q-1)}(0) = E_r^{(2q-1)}(1) = 0 \qquad \text{for} \quad 1 \leqslant q \leqslant (r-1)/2 \tag{16b}$$

provided that r is odd.

Additionally, E_r is with respect to the line $x = 1/2$ an even or an odd function if r is even or odd, respectively. Hence the perfect Euler spline F defined by

$$F(x) = (-1)^j (n+1)^{-r} E_r((n+1)x - j) \qquad \text{for} \quad x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], \tag{17}$$

$0 \leqslant j \leqslant n$, has the following properties. $F \in W_r \subset W_{r,1,\Delta^*}$, F interpolates the zero function in the sense of (13), and

$$F^{(r)}(x) = (-1)^j \qquad \text{for} \quad x \in \left(\frac{j}{n+1}, \frac{j+1}{n+1}\right), \qquad 0 \leqslant j \leqslant n. \tag{18}$$

In other words, the perfect Euler spline F is the extremal function of Theorem 2 for the interpolation problem (13) and the weight function $w \equiv 1$. Hence we have proved

THEOREM 3. *Let Δ^* be the collection* (12) *of equidistant knots. For each function $f \in W_r$ there exists a unique spline interpolant $s_f \in \mathrm{Sp}(r, \Delta^*)$ which interpolates f in the sense of* (13). *Moreover,*

$$|f(x) - s_f(x)| \leqslant (n+1)^{-r} E_r((n+1)x - j) \qquad \textit{for} \quad x \in \left[\frac{j}{n+1}, \frac{j+1}{n+1}\right], \tag{19}$$

$0 \leqslant j \leqslant n$, *and*

$$\sup\{\| f - s_f \| : f \in W_r\} = (n+1)^{-r} \| E_r \| = (n+1)^{-r} K_r \pi^{-r}. \tag{20}$$

To illustrate the extremely good approximation properties of interpolating polynomial splines with equidistant knots we have to introduce the concept of the n-widths.

DEFINITION 4. If $W \subset C[0, 1]$, then the n-width of W in $C[0, 1]$ is defined to be

$$d_n(W) = \inf_{X_n} \sup_{f \in W} \inf_{g \in X_n} \| f - g \| \tag{21}$$

where the infimum is taken over all n-dimensional linear subspaces X_n of $C[0, 1]$.

If, for instance, W_r^* is the restriction of the class of functions

$$\{f \mid f^{(r-1)} \text{ is absolutely continuous on the real line, } f \text{ has the period } 1, |f^{(r)}(x)| \leqslant 1 \text{ a.e.}\}$$

to the interval $[0, 1]$, it is well known (see Tihomirov [10]) that

$$d_{2n-1}(W_r^*) = d_{2n}(W_r^*) = K_r \pi^{-r}(2n)^{-r}. \tag{22}$$

Since $W_r^* \subset W_r$, it follows from (22) that

$$d_n(W_r) \geqslant d_n(W_r^*) \geqslant K_r \pi^{-r}(n+1)^{-r} \tag{23}$$

and from (20) and (23) that

$$\sup\{\|f - s_f\| : f \in W_r\} \leqslant d_n(W_r). \tag{24}$$

Since $\dim \operatorname{Sp}(r, \Delta^*) = n + r$ we should also look at d_{n+r}. From (20) and (23) we obtain

$$d_{n+r}(W_r) \leqslant \sup\{\|f - s_f\| : f \in W_r\} \leqslant \left(\frac{n+r+1}{n+1}\right)^r d_{n+r}(W_r) \tag{25}$$

and

$$d_n(W_r) = K_r \pi^{-r}(n+1)^{-r}(1 + O(n^{-1})) \qquad \text{as} \quad n \to \infty. \tag{26}$$

Thus we have shown that the interpolation (13) by splines of degree $r - 1$ with equidistant knots leads to asymptotically best possible error bounds in the sense of n-widths.

Remark 2. It has been shown by Tihomirov [10] (see also Chui and Smith [1], Micchelli and Pinkus [7, 8]) that there exists a unique collection Δ_{nr} of $n - r$ knots in $(0, 1)$,

$$\Delta_{nr}: 0 = t_0 < t_1 < \cdots < t_{n-r} < t_{n+1-r} = 1, \qquad n > r,$$

(depending on n and r and not equidistant) such that the spline subspace $\operatorname{Sp}(r, \Delta_{nr})$ of dimension n is an extremal subspace for the class W_r in the sense of n-widths. Again, interpolation at certain n points achieves the desired order of magnitude $d_n(W_r)$ of the error. (See Micchelli and Pinkus [7, 8] for more details and more general results.)

Comparing (20) and (26) we conclude that interpolation by splines with *equidistant* knots is easier to handle than by the extremal spline subspace $\operatorname{Sp}(r, \Delta_{nr})$, but the errors are essentially the same, at least for the class W_r.

Remark 3. Inequality (20) in Theorem 3 is related to results of C. A. Micchelli and A. Pinkus [7; page 169, Example 4] where for the case r even it is done in greater generality. Inequality (20) has also been established by Hall and Meyer [3] for even r. Hall and Meyer do not mention the connection to n-widths, but for cubic splines they also establish excellent bounds for the derivatives error which I want to analyze here using n-widths.

THEOREM 4 (Hall and Meyer [3, Theorem 5]). *Let Δ be an arbitrary collection* (2) *of knots. Let $s_f \in \mathrm{Sp}(4, \Delta)$ be the cubic spline interpolant of $f \in C^4[0, 1]$ with respect to the interpolation problem* (13a) *or the interpolation problem*

$$s(t_j) = f(t_j) \quad \text{for } 0 \leqslant j \leqslant n+1, \quad s'(0) = f'(0), \quad s'(1) = f'(1). \tag{27}$$

Then,

$$\| f^{(q)} - s_f^{(q)} \| \leqslant C_q \| f^{(4)} \| h^{4-q} \qquad \text{for} \quad q = 0, 1, 2, 3 \tag{28}$$

where $C_0 = 5/384$, $C_1 = 1/24$, $C_2 = 3/8$, $C_3 = (\eta + \eta^{-1})/2$ *and*

$$h = \max_{0 \leqslant j \leqslant n} h_j, \qquad h_j = t_{j+1} - t_j, \qquad \eta = h / \min_{0 \leqslant j \leqslant n} h_j. \tag{29}$$

The constants C_0 and C_1 are optimal in the sense that

$$C_q = \sup \left\{ \frac{\| f^{(q)} - s_f^{(q)} \|}{\| f^{(4)} \| h^{4-q}} : f \in C^4[0, 1], f^{(4)} \not\equiv 0, \Delta \text{ arbitrary} \right\}$$

for $q = 0$, $q = 1$.

An easy calculation shows that

$$C_0 = \| E_4 \| = K_4 \pi^{-4}, \quad C_1 = \| E_3 \| = K_3 \pi^{-3}, \quad C_2 = 3 \| E_2 \| = 3K_2 \pi^{-2}. \tag{30}$$

Therefore, in the equidistant case $h = 1/(n+1)$ the constant C_0 in (28) is the same as in (20), and the inequalities (28) for $q = 1$ and (23) for $r = 3$ lead to the surprising result that

$$d_{n+3}(C_1{}^3[0, 1]) \leqslant \sup\{\| f' - s_f' \| : f \in C_1{}^4[0, 1]\} \leqslant d_n(C_1{}^3[0, 1]) \tag{31}$$

where $C_1{}^r[0, 1] := \{f \in C^r[0, 1] : \| f^{(r)} \| \leqslant 1\}$, and we have applied that $C_1{}^r[0, 1]$ is dense in W_r and hence

$$d_n(C_1{}^r[0, 1]) = d_n(W_r), \qquad C_1{}^3[0, 1] = \{f' \mid f \in C_1{}^4[0, 1]\},$$

$s_f' \in \mathrm{Sp}(3, \Delta)$, and $\dim \mathrm{Sp}(3, \Delta) = n + 3$.

Consequently, interpolation by cubic splines with equidistant knots in the sense of (13a) or (27) is up to a negligible factor best possible in the sense of n-widths not only for the functions itself but also for their first derivatives. It is as far as I know an open problem if the same holds for higher derivatives and for splines of higher degree.

5. Local Upper Bounds in Cubic Spline Interpolation

In this section we discuss a second special case where we also can construct the extremal function F of Theorem 2 explicitly. We discuss the following interpolation problem: For $r = 4$, $n \geqslant 0$, and an arbitrary collection (2) of knots $\Delta: 0 = t_0 < t_1 < \cdots < t_{n+1} = 1$ let $s_f \in \mathrm{Sp}(4, \Delta)$ be the unique cubic spline interpolant of $f \in C^2[0, 1]$ such that

$$s_f(t_j) = f(t_j) \quad \text{for } 0 \leqslant j \leqslant n+1, \quad s_f''(0) = f''(0), \quad s_f''(1) = f''(1). \tag{32}$$

It is easy to verify that the function

$$F(x) = (-1)^j \, h_j h^3 E_4\left(\frac{x - t_j}{h_j}\right) \quad \text{for } x \in [t_j, t_{j+1}], \quad 0 \leqslant j \leqslant n, \tag{33}$$

is the extremal function of Theorem 2 for the interpolation problem (32) and the weight function w,

$$w(x) = (h/h_j)^3 \qquad \text{for} \quad x \in (t_j, t_{j+1}), \qquad 0 \leqslant j \leqslant n, \tag{34}$$

where E_4 is the Euler polynomial of degree 4 and h and h_j are defined in (29). Hence, by Theorem 2, we have proved

THEOREM 5. *For any function $f \in W_{4,w,\Delta}$,*

$$W_{4,w,\Delta} = \{f \in C^2[0, 1] \mid f^{(3)} \text{ is absolutely continuous in each } (t_j, t_{j+1}), \text{ and } |f^{(4)}(x)| \leqslant (h/h_j)^3 \text{ a.e. in } (t_j, t_{j+1}), 0 \leqslant j \leqslant n\},$$

the unique cubic spline interpolant s_f of (32) *satisfies the following inequalities*:

$$|f(x) - s_f(x)| \leqslant h_j h^3 E_4\left(\frac{x - t_j}{h_j}\right), \quad x \in [t_j, t_{j+1}], \quad 0 \leqslant j \leqslant n, \tag{35}$$

$$\|f - s_f\| \leqslant K_4 \pi^{-4} h^4, \tag{36}$$

$$|f'(t_j) - s_f'(t_j)| \leqslant K_3 \pi^{-3} h^3, \qquad 0 \leqslant j \leqslant n+1. \tag{37}$$

If we compare Theorem 5 with the results of Hall and Meyer in Theorem 4, we realize that we could not establish error bounds for the derivatives; however we obtained the inequality (28) for $q = 0$ (remember (30)) even for the larger class of functions $W_{4,w,\Delta}$. Furthermore, inequality (35) yields best possible local error bounds which seem to be new.

Remark 4. The results and the proofs of Theorems 2, 3, and 5 can easily be extended to the interpolation of periodic functions by periodic splines.

ACKNOWLEDGMENT

The author has written the greater part of this paper at the University of Texas at Austin. He is greatly indebted to Professor E. W. Cheney for his invitation and for many helpful discussions.

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